International Journal of Solids and Structures 51 (2014) 1809–1818

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Stress gradient versus strain gradient constitutive models within elasticity

Castrenze Polizzotto *

Università di Palermo, Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, dei Materiali, Viale delle Scienze, 90128 Palermo, Italy

ARTICLE INFO

Article history:
Received 5 October 2013
Received in revised form 29 November 2013
Available online 4 February 2014

Keywords:
Stress gradient elasticity
Principle of the virtual power
Boundary conditions
Beam models

ABSTRACT

A stress gradient elasticity theory is developed which is based on the Eringen method to address nonlocal elasticity by means of differential equations. By suitable thermodynamics arguments (involving the free enthalpy instead of the free internal energy), the restrictions on the related constitutive equations are determined, which include the well-known Eringen stress gradient constitutive equations, as well as the associated (so far uncertain) boundary conditions. The proposed theory exhibits complementary characters with respect to the analogous strain gradient elasticity theory. The associated boundary-value problem is shown to admit a unique solution characterized by a Hellinger–Reissner type variational principle. The main differences between the Eringen stress gradient model and the concomitant Aifantis strain gradient model are pointed out. A rigorous formulation of the stress gradient Euler–Bernoulli beam is provided; the response of this beam model is discussed as for its sensitivity to the stress gradient effects and compared with the analogous strain gradient beam model.

1. Introduction

Eringen (1983) proposed a method to address boundary-value problems of nonlocal (integral) elasticity whereby the inherent integro-differential equations are replaced by differential equations. The method is grounded on the constitutive relation

$$\underbrace{\mathbf{C}:\boldsymbol{\varepsilon}}_{\mathbf{s}} = \underbrace{\boldsymbol{\sigma} - \ell^2 \Delta \boldsymbol{\sigma}}_{\mathrm{L}\boldsymbol{\sigma}} \tag{1}$$

where ℓ is a material constant with the meaning of internal length scale parameter, Δ is the Laplacian operator and $\mathbf{C}$ is the usual fourth order moduli tensor of isotropic elasticity. With the language of nonlocal elasticity, (1) can be qualified as a differential relation (featured by the operator $L := 1 - \ell^2\Delta$) between the *nonlocal* stress field $\boldsymbol{\sigma}$ and the *local* strain field $\boldsymbol{\varepsilon}$ (or the associated *local* Hookean stress $\mathbf{s} := \mathbf{C} : \boldsymbol{\varepsilon}$), that is, between two fields which more naturally are related through an integral-type relation as

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|)\mathbf{s}(\mathbf{x}')\, \mathrm{d}v(\mathbf{x}') \tag{2}$$

Here, V is the material domain and $\alpha(|\mathbf{x}' - \mathbf{x}|)$ is the *influence function* (Eringen, 2002). This is a positive function of the distance $|\mathbf{x}' - \mathbf{x}|$ between the field point $\mathbf{x}$ and the source point $\mathbf{x}'$; it has a maximum value at $\mathbf{x}' = \mathbf{x}$ and decays more or less rapidly with the increasing distance $|\mathbf{x}' - \mathbf{x}|$, becoming vanishing at all points $\mathbf{x}'$ located out of a sphere of (relatively small) radius R and centered at $\mathbf{x}$. The equivalence between (1) and (2) stems from the restriction that α is the Green function of the operator L. In fact, on applying the latter operator to (2), since $\mathrm{L}\alpha = \delta_\mathrm{D} =$ Dirac delta, (1) can be readily obtained.

The nonlocal stress field $\boldsymbol{\sigma}$ is required to satisfy the standard equilibrium equations, namely,

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V, \qquad \mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on } S_f \tag{3}$$

where $\mathbf{b}$ denotes body forces within V and $\bar{\mathbf{t}}$ surface forces assigned over the free part S_f of the boundary surface $S = \partial V$; $\mathbf{n}$ is the unit outward normal to S. The latter body forces are presumed to include the inertia forces, if any. Furthermore, the local strain field $\boldsymbol{\varepsilon}$ is required to satisfy the standard compatibility equations, that is,

$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} \quad \text{in } V, \qquad \mathbf{u} = \bar{\mathbf{u}} \quad \text{on } S_c \tag{4}$$

where ∇^s denotes the symmetric part of the gradient operator ∇, whereas $\bar{\mathbf{u}}$ is the imposed displacement on the constrained part $S_c = S \setminus S_f$ of S.

The Eringen method consists in associating (3) and (4) with the differential constitutive equations of (1) instead of the integral type ones of (2). This means that the *non-locality effects* of the original integral-type problem enter into play within the

* Tel.: +39 091 23896718.
E-mail addresses: castrenze.polizzotto@unipa.it, c.polizzotto@inwind.it

http://dx.doi.org/10.1016/j.ijsolstr.2014.01.021
0020-7683/

differential-type problem as *gradient effects* originating from a source identified with the Cauchy stress $\boldsymbol{\sigma}$. In other words, the original nonlocal integral-type model is replaced with a *stress gradient model*. Indeed the latter model finds itself in strong contrast with the well-known *strain gradient model* widely employed to describe size effects and other phenomena of small scale solids.

The popularity of the above Eringen method stems from the relative easiness with which a differential type boundary-value problem can be solved with respect to one of integro-differential nature. Indeed, on combining (1), (3) and (4), one easily obtains the following displacement equation

$$\mathcal{L}\mathbf{u} = -\mathbf{b}^* \quad \text{in } V \tag{5}$$

where

$$\mathbf{b}^* := \mathrm{L}\mathbf{b} = \mathbf{b} - \ell^2\Delta\mathbf{b} \tag{6}$$

The symbol $\mathcal{L}$ denotes the classical set of second-order partial differential equations (PDEs) of isotropic elasticity, that is, denoting by λ and μ the Lamé constants,

$$\mathcal{L}\mathbf{u} := \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\nabla\cdot\mathbf{u}. \tag{7}$$

The PDE system (5) has to be solved in association with the boundary conditions $(3)_2$ and $(4)_2$, of which the former carries in the stress $\boldsymbol{\sigma}$. This implies that the obtained boundary-value problem cannot in general be solved without considering the PDEs (1) together with the associated boundary conditions (heuristically devised, since their exact form is unknown from the wide literature, to the author's knowledge).

The above method has been widely used to address problems within nanotechnology, crack problems at the microscale, dislocation analysis within unbounded domains, etc. For more complex problems an approximation can be taken by replacing the coupling boundary condition $(3)_2$ with a similar uncoupling one in which the nonlocal stress $\boldsymbol{\sigma}$ is replaced by the local stress $\mathbf{s}$. In this way the resulting boundary-value problem identifies—except for the body force, if any—with the classical one, whereas the related stress field $\boldsymbol{\sigma}$ may then be obtained by solving the PDEs (1), (indeed, an operation which requires due care since there cannot be any guarantee that the derived stress field satisfies the equilibrium equations). It is not the purpose of the present paper to review the extensive literature on this topic; we just mention some representative works and the references therein, namely Eringen (1983, 2002), Lazar et al. (2006a,b), Askes and Gutiérrez (2006), Reddy (2007), Reddy and Pang (2008), Peddieson et al. (2003), Kumar et al. (2008).

The present paper is more interested in other aspects of the Eringen method, emerging when the stress gradient model discussed previously is compared with a strain gradient model featured by a constitutive equation similar to (1), that is,

$$\boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \ell^2\Delta\boldsymbol{\varepsilon}) \tag{8}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress satisfying the equilibrium equations (3) and $\boldsymbol{\varepsilon}$ is the standard strain satisfying the compatibility equations (4). The gradient elasticity model based on (8)—often referred to as the Aifantis elasticity model (Aifantis, 1992; Ru and Aifantis, 1993; Altan and Aifantis, 1997)—can be viewed as a particularization of a more general strain gradient model devised by Mindlin (1965), Mindlin and Eshel (1968), Wu (1992); see Askes and Aifantis (2011) for an overview on the latter models. On comparing (8) and (1) with each other, one can observe that the stress gradient model (1) exhibits a character of *complementarity* (in the mechanical sense) with respect to the strain gradient model (8). However, whereas the thermodynamic consistency of (8) as a gradient constitutive model has been already assessed within the literature (see e.g., Polizzotto, 2011 and the literature therein), no such investigations seem to exist for (1). It is therefore quite natural to raise the following question:

Is there any thermodynamics-based procedure which, like the analogous procedures devised for the strain gradient models, may lead to the Eringen constitutive equation (1) and to the related boundary conditions?

The main purpose of the present paper is to give a positive answer to the latter question. Indeed, it will be shown that any thermodynamics-based procedure devised for a strain gradient model, but suitably changed into one of complementary nature, may constitute a procedure suitable to cope with a stress gradient model. This requires that (i) the principle of the virtual power (PVP) (for velocities) must be replaced with the complementary PVP (for stress rates), and (ii) the internal energy and the (Helmholtz) free energy must be replaced with the enthalpy and the (Gibbs) free enthalpy, respectively.

The outline of the paper is as follows. In Section 2, some thermodynamics premises are developed in the purpose to obtain a complementary form of the Clausius–Duhem inequality expressed in terms of the Gibbs function, that is, a thermodynamic potential depending on the stress, the temperature and, possibly, the stress gradient. In Section 3, an extended form of the principle of the virtual power (PVP) for stress gradient materials is presented, which is the complementary counterpart of the analogous PVP for strain gradient materials well-known from the literature (Mindlin, 1965; Germain, 1973), and which leads to the higher order compatibility equations. In Section 4, the results derived in the preceding sections are used to determine the restrictions on the constitutive equations for a stress gradient material; as an alternative to the PVP, a complementary form of the so-called energy residual can be used. The obtained restrictions include the constitutive equations (coinciding with the Eringen stress differential equations (1)), as well as the related boundary conditions in the form $\partial_n\sigma_{ij} = 0$ at all points of the boundary surface. In Section 5 the boundary-value problem associated to the Eringen stress gradient model is addressed and shown to admit a unique solution characterized by two variational principles. One of the latter principles is a minimum principle for the problem to evaluate the stress field associated to a specified strain field through the gradient stress–strain relation and related boundary conditions; the other is a Hellinger–Reissner type principle for the stress and displacement response of a structure subjected to given loads. For comparison purposes, the boundary-value problem associated to a particular class of strain gradient materials (Aifantis elasticity model) is addressed in Section 6, pointing out the main differences between the two models. In Section 7 the Hellinger–Reissner principle is used to derive a complete theory for the Euler–Bernoulli beam, which is discussed in contrast to the analogous strain gradient beam model. Conclusions are drawn in Section 8.

Notation. A compact notation is used, with boldface letters denoting vectors or tensors of any order. The scalar product between vectors or tensors is denoted with as many dots as the number of contracted index pairs. For instance, denoting by $\mathbf{u} = \{u_i\}, \mathbf{v} = \{v_i\}$, $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}, \boldsymbol{\sigma} = \{\sigma_{ij}\}, \boldsymbol{\tau} = \{\tau_{ijk}\}$ and $\mathbf{A} = \{A_{ijkh}\}$ some vectors and tensors, one can write: $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$, $\mathbf{A} : \boldsymbol{\varepsilon} = \{A_{ijkh}\varepsilon_{kh}\}$, $\mathbf{A}\,\vdots\,\boldsymbol{\tau} = \{A_{ijkh}\tau_{jkh}\}$, $\mathbf{A}^T\vdots\,\boldsymbol{\tau} = \{A_{ijkh}\tau_{kji}\}$. The summation rule for repeated indexes holds and the subscripts denote components with respect to an orthogonal Cartesian co-ordinate system, say $\mathbf{x} = (x_1, x_2, x_3)$. The tensor product is simply indicated as, for instance, $\mathbf{uv} = \{u_i v_j\}$, and thus $\mathbf{A} : \mathbf{uv} = \{A_{ijkh}u_k v_h\}$. An upper dot over a symbol denotes its (material) time derivative, $\dot{\mathbf{u}} = \mathrm{d}\mathbf{u}/\mathrm{d}t$. The symbol ∇ denotes the spatial gradient operator, i.e., $\nabla\mathbf{u} = \{\partial_i u_j\}$, ∇^s is the symmetric part of ∇, and Δ is the Laplacian operator. The symbol $:=$ means equality by definition. Other symbols will be defined in the text at their first appearance.

2. Thermodynamics premises

For a simple material, the internal energy balance equation (or first thermodynamics principle) reads as (Lemaitre and Chaboche, 1990; Maugin, 1999):

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + h - \nabla \cdot \mathbf{q} \tag{9}$$

where $u = u(\boldsymbol{\varepsilon}, N)$ is the internal energy (per unit volume), h is the radiation heat supply, $\mathbf{q}$ is the heat flux and N the entropy. Within the framework of solid mechanics, in which the strains and the (absolute) temperature $T > 0$ play the role of driving variables, the Helmholtz free energy $\psi = \psi(\boldsymbol{\varepsilon}, T)$ is preferably used in place of u, to which it is related through the Legendre relation

$$u = NT + \psi \tag{10}$$

However, in the case whereby the constitutive equation involves the stress and (possibly) the temperature, a most appropriate thermodynamic potential function is the Gibbs function, say $G(\boldsymbol{\sigma}, T)$, which is related to u by the Legendre relation

$$u = NT + \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - G \tag{11}$$

Therefore, on differentiating (11) with respect to time and then substituting the obtained expression of $\dot{u}$ into (9) gives the equality

$$N\dot{T} + T\dot{N} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} - \dot{G} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + h - \nabla \cdot \mathbf{q} \tag{12}$$

Next, let us consider the internal entropy production, say γ, and let us write the inherent inequality (or second thermodynamics principle), that is,

$$\gamma := \dot{N} - \frac{h}{T} + \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) \geqslant 0 \tag{13}$$

Then, solving (12) for $\dot{N}$ and substituting the obtained expression into (13), we have the inequality

$$T\gamma = \underbrace{-(\boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} - \dot{G} - N\dot{T})}_{\Phi} + \underbrace{\left(-\frac{\mathbf{q}}{T} \cdot \nabla T\right)}_{\Phi_{\mathrm{th}}} \geqslant 0 \tag{14}$$

Here, the quantity Φ_{th} defined as

$$\Phi_{\mathrm{th}} := -\frac{\mathbf{q}}{T} \cdot \nabla T \geqslant 0 \tag{15}$$

is the *thermal dissipation by conduction* (usually assumed nonnegative by its own), whereas the quantity $-\Phi$ defined as

$$-\Phi := \underbrace{\boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}}}_{\mathcal{W}^c} - \dot{G} + N\dot{T} \leqslant 0 \tag{16}$$

is the *negative intrinsic dissipation*. Inequality (16) represents a complementary form of the Clausius–Duhem inequality expressed in terms of the Gibbs function. It is formally written for a simple material, but it holds also for a stress gradient one, provided that the strain power $\mathcal{W}^c = \boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}}$ be complemented by considering the contribution from the stress gradient rate(s), and the arguments of the Gibbs function be enriched by the pertinent stress gradient variables, as it will be explained in details shortly.

3. The complementary PVP. Extended form for stress gradient elasticity

Let us consider an elastic material featured by *intrinsic* stress modes (that is, independent loading modes to which the material is reactive by responding with specific deformation modes of its own), which consist of the classical Cauchy stress, say $\boldsymbol{\sigma}$, and its first gradient, $\nabla\boldsymbol{\sigma}$. Then, the *strain power*, that is, the complementary power expended by the surrounding material on the generic particle per unit volume, can be expressed as

$$\mathcal{W}^c = \mathbf{e} : \dot{\boldsymbol{\sigma}} + \boldsymbol{\eta} \,\vdots\, \nabla\dot{\boldsymbol{\sigma}} \tag{17}$$

where the strain tensors $\mathbf{e} = \{e_{ij}\}$ and $\boldsymbol{\eta} = \{\eta_{kij}\}$, both symmetric with respect to the index pair (i,j), are the "intrinsic" strains power-conjugate to $\dot{\boldsymbol{\sigma}} = \{\dot{\sigma}_{ij}\}$ and $\nabla\dot{\boldsymbol{\sigma}} = \{\partial_k \dot{\sigma}_{ij}\}$, respectively. A material like the above is qualified as a *stress gradient material*, a name which indeed mimics the name of "strain gradient material" well known from the literature.

Next, let us consider a stress gradient material occupying the domain V with boundary surface $S = \partial V$, which is subjected to body forces $\mathbf{b}$ within V, tractions $\bar{\mathbf{t}}$ on $S_f \subset S$, and imposed displacements $\bar{\mathbf{u}}$ on $S_c = S \setminus S_f$. The Cauchy stress $\boldsymbol{\sigma}$ is in equilibrium with the given loads, hence we can write the equations (3), here reported again for more clarity, that is

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V, \qquad \mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on } S_f. \tag{18}$$

As a consequence, the kinematic variables $(\mathbf{e}, \boldsymbol{\eta})$ are expected to be dependent on each other, but we do not know how they may be mutually related. For this purpose an extended form of the (classical) complementary PVP can be applied, which is the counterpart of the analogous extended form of the (primary) PVP mentioned in the Introduction.

Let us consider a real deformation process of the material and let us make reference to a time at which the stress and strain states of the material are known. With the complementary PVP, some *virtual stresses* are applied on the loaded solid in order to artificially modify its stress state, but maintaining the equilibrium conditions, whereas its deformation state is taken fixed. Then, denoting by a superposed tilde the virtual mechanical variables, the complementary PVP can be cast in the following form:

$$\int_B (\mathbf{e} : \tilde{\boldsymbol{\sigma}} + \boldsymbol{\eta} \,\vdots\, \nabla\tilde{\boldsymbol{\sigma}})\, \mathrm{d}v = \int_{\partial B} \mathbf{r} : \tilde{\boldsymbol{\sigma}}\, \mathrm{d}a \tag{19}$$

where B is any subdomain of V, whereas $\mathbf{r} = \{r_{ij}\}$ denotes a second order strain tensor working through $\tilde{\boldsymbol{\sigma}}$ on the boundary surface ∂B. The virtual stresses $\tilde{\boldsymbol{\sigma}}$ are arbitrary, but satisfy the equilibrium equations (18) in homogeneous form, that is:

$$\nabla \cdot \tilde{\boldsymbol{\sigma}} = \mathbf{0} \quad \text{in } B, \qquad \mathbf{n} \cdot \tilde{\boldsymbol{\sigma}} = \mathbf{0} \quad \text{on } \partial B, \qquad \forall B \subseteq V \tag{20}$$

Therefore, denoting by $\mathbf{u}$ a displacement Lagrange multiplier defined within $B \cup \partial B$, we can append the constraints (20) to (19) which thus takes on the form

$$\int_B (\mathbf{e} : \tilde{\boldsymbol{\sigma}} + \boldsymbol{\eta} \,\vdots\, \nabla\tilde{\boldsymbol{\sigma}})\, \mathrm{d}v + \int_B \nabla \cdot \tilde{\boldsymbol{\sigma}} \cdot \mathbf{u}\, \mathrm{d}v - \int_{\partial B} \mathbf{r} : \tilde{\boldsymbol{\sigma}}\, \mathrm{d}a - \int_{\partial B} \mathbf{n} \cdot \tilde{\boldsymbol{\sigma}} \cdot \mathbf{u}\, \mathrm{d}a = 0 \tag{21}$$

Next, by the divergence theorem, the latter equality can be rewritten as follows

$$\int_B (\underbrace{\mathbf{e} - \nabla \cdot \boldsymbol{\eta}}_{\boldsymbol{\varepsilon}} - \nabla^s \mathbf{u}) : \tilde{\boldsymbol{\sigma}}\, \mathrm{d}v + \int_{\partial B} (\mathbf{n} \cdot \boldsymbol{\eta} - \mathbf{r}) : \tilde{\boldsymbol{\sigma}}\, \mathrm{d}a = 0 \tag{22}$$

As this equality has to hold for arbitrary choices of $\tilde{\boldsymbol{\sigma}}$ within $B \cup \partial B$, we obtain

$$\left.\begin{array}{ll} \boldsymbol{\varepsilon} := \mathbf{e} - \nabla \cdot \boldsymbol{\eta} = \nabla^s \mathbf{u} & \text{in } B \\ \mathbf{n} \cdot \boldsymbol{\eta} = \mathbf{r} & \text{on } \partial B \end{array}\right\} \quad \forall B \subseteq V \tag{23}$$

These equations are the desired *compatibility equations* in which the compatible strain $\boldsymbol{\varepsilon}$ is found as the *total strain* derived from the intrinsic deformation modes $(\mathbf{e}, \boldsymbol{\eta})$. Eq. $(23)_2$ provides the *kinematic higher order boundary conditions*.

4. Restrictions on the constitutive equations

The restrictions on the constitutive equations for the stress gradient model under consideration can be assessed by the complementary Clausius–Duhem inequality (16). For sake of simplicity, isothermal conditions are assumed, hence the Gibbs function can be considered independent of the temperature T.

4.1. By the complementary PVP

Within the present context of stress gradient constitutive behavior, the Gibbs function has to be considered as a function of the stress and of its first gradient, say $G = G(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})$, whereas the complementary strain power $\mathcal{W}^c$ takes on the form (17). Therefore, inequality (16) has to be rewritten as

$$-\Phi = \mathbf{e} : \dot{\boldsymbol{\sigma}} + \boldsymbol{\eta} \vdots \nabla\dot{\boldsymbol{\sigma}} - \overline{G(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})}^{\cdot} \leqslant 0 \tag{24}$$

which holds for any real deformation process of the material. Expanding the time derivative of G, we obtain from (24)

$$\left(\mathbf{e} - \frac{\partial G}{\partial \boldsymbol{\sigma}}\right) : \dot{\boldsymbol{\sigma}} + \left(\boldsymbol{\eta} - \frac{\partial G}{\partial(\nabla\boldsymbol{\sigma})}\right) \vdots \nabla\dot{\boldsymbol{\sigma}} \leqslant 0 \tag{25}$$

which, considering that the intrinsic strains $\mathbf{e}, \boldsymbol{\eta}$ are independent of $\dot{\boldsymbol{\sigma}}$ and $\nabla\dot{\boldsymbol{\sigma}}$, can be satisfied if, and only if,

$$\mathbf{e} = \frac{\partial G}{\partial \boldsymbol{\sigma}}, \quad \boldsymbol{\eta} = \frac{\partial G}{\partial(\nabla\boldsymbol{\sigma})} \tag{26}$$

These are the state equations of the intrinsic (or "primitive") strains, which thus prove to be some functions of the stress and the stress gradient. Then, the state equation of the total strain $\boldsymbol{\varepsilon}$ of $(23)_1$ is given by

$$\boldsymbol{\varepsilon} = \frac{\partial G}{\partial \boldsymbol{\sigma}} - \nabla \cdot \left(\frac{\partial G}{\partial(\nabla\boldsymbol{\sigma})}\right) \tag{27}$$

A simple choice for G may be as follows

$$G = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{A} : \boldsymbol{\sigma} + \frac{1}{2}\ell^2 \mathbf{A} :: \left[(\nabla\boldsymbol{\sigma})^{\mathrm{T}} \cdot \nabla\boldsymbol{\sigma}\right] \tag{28}$$

where $\mathbf{A} = \{A_{ijkl}\}$ is the usual compliance tensor of isotropic elasticity ($\mathbf{A} = \mathbf{C}^{-1}$), whereas ℓ is a material length scale parameter. Then, we have

$$\mathbf{e} = \mathbf{A} : \boldsymbol{\sigma}, \quad \boldsymbol{\eta} = \ell^2 \nabla \mathbf{e} \tag{29}$$

and

$$\boldsymbol{\varepsilon} = \mathbf{A} : (\boldsymbol{\sigma} - \ell^2 \Delta\boldsymbol{\sigma}) \tag{30}$$

Indeed, we can observe that (30) coincides with the Eringen stress gradient model (1) and that it is cast as a complementary version of the Aifantis strain gradient elasticity model, which reads

$$\boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \ell^2 \Delta\boldsymbol{\varepsilon}) \tag{31}$$

where $\boldsymbol{\varepsilon}$ is just a compatible strain field. We also observe that the Eringen stress formula (30) is here derived on an energetic basis centered on the chosen Gibbs function (28), which is in contrast to (1) obtained as an inversion of the integral model by Eringen (1983) and, more recently, by Lazar et al. (2006a,b).

4.2. By the complementary energy residual

It is possible to avoid the involvement of the complementary PVP by the use of the energy residual (Polizzotto, 2011; Polizzotto and Pisano, 2012), but cast in its complementary form, that is the scalar quantity

$$P^c = \nabla \cdot (\boldsymbol{\eta} : \dot{\boldsymbol{\sigma}}) \tag{32}$$

Let us observe that the energy residual P^c exhibits its typical divergence format, but here the source of gradient effects is played by the stress rate, whereas the concomitant higher order tensor is a third-order strain tensor $\boldsymbol{\eta}$.

The complementary energy residual P^c intervenes into the complementary Clausius–Duhem inequality, which has to be written by considering the strain potential in its classical format, that is

$$-\Phi = \underbrace{\boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} + P^c}_{\mathcal{W}^c} - \overline{G(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})}^{\cdot} \leqslant 0 \tag{33}$$

where the symbol $\boldsymbol{\varepsilon}$ has been used in place of $\mathbf{e}$ just for formal convenience. Expanding the time derivative of G and the divergence operator of P^c gives

$$-\Phi = \left(\boldsymbol{\varepsilon} + \nabla \cdot \boldsymbol{\eta} - \frac{\partial G}{\partial \boldsymbol{\sigma}}\right) : \dot{\boldsymbol{\sigma}} + \left(\boldsymbol{\eta} - \frac{\partial G}{\partial(\nabla\boldsymbol{\sigma})}\right) \vdots \nabla\dot{\boldsymbol{\sigma}} \leqslant 0 \tag{34}$$

from where we can obtain substantially the same results as with the PVP, that is

$$\boldsymbol{\eta} = \frac{\partial G}{\partial(\nabla\boldsymbol{\sigma})}, \quad \boldsymbol{\varepsilon} = \underbrace{\frac{\partial G}{\partial \boldsymbol{\sigma}}}_{\mathbf{e}} - \nabla \cdot \underbrace{\left(\frac{\partial G}{\partial(\nabla\boldsymbol{\sigma})}\right)}_{\boldsymbol{\eta}} \tag{35}$$

4.3. Boundary conditions for the stress gradient elasticity model

Eq. (27), and in particular (30), provides the constitutive elasticity laws for the considered stress gradient material model, shaped as a set of second-order PDEs. These equations can be integrated to obtain the stress field $\boldsymbol{\sigma}$ in terms of the total strain $\boldsymbol{\varepsilon}$ assumed known within the domain V. The boundary conditions necessary to this purpose can be obtained by the PVP enforced for the whole domain. Then, looking at (22), and in particular to the last surface integral, we can write, recalling $(35)_1$, the boundary conditions as

$$\mathbf{n} \cdot \boldsymbol{\eta} = \mathbf{n} \cdot \left(\frac{\partial G}{\partial(\nabla\boldsymbol{\sigma})}\right) = \mathbf{r} \quad \text{on } S \tag{36}$$

The tensor $\mathbf{r}$ interprets the influence of the external environment on the constitutive behavior of the material; as a matter of facts, its evaluation is likely difficult in practice. Assuming $\mathbf{r} \equiv \mathbf{0}$ (which amounts to assuming the body to be a constitutively closed system, as indeed it is the case in practice), for the Eringen model featured by (29) and (30) the boundary condition (36) takes the notable form

$$\ell^2 \mathbf{A} : \partial_n \boldsymbol{\sigma} = \mathbf{0} \quad \rightarrow \quad \partial_n \boldsymbol{\sigma} = \mathbf{0} \quad \text{on } S \tag{37}$$

Namely, the normal derivatives of all the Cauchy stress components must vanish at all points of the boundary surface S. This is indeed a peculiar restriction due to the stress gradient effects.

An insight on the physical meaning of the latter boundary conditions can be achieved by referring the stresses σ_{ij} at any point $\mathbf{x} \in S$ to local orthogonal exes, say (x_1, x_2, x_3), with x_3 directed as the normal $\mathbf{n}$. The conditions $\sigma_{ij,3} = 0$ on S imply that every stress component saves a constant value at any point close to S and moving toward the surface in the normal direction. This in turn implies that the continuum equilibrium equations at a point within the bulk material, at the limit when the point approaches to the boundary surface S along a normal direction, tend to take on the shape of surface equilibrium equations for a thin membrane-like boundary layer circumventing the material. Indeed, this boundary layer finds itself in equilibrium under through-thickness constant stresses, whereas it entirely transmits to the adjacent bulk material the tractions applied on its external face.

5. The boundary-value problem for stress gradient elasticity

Having in mind the Eringen stress gradient model previously discussed, let us lay down the related boundary-value problem. The equations governing the latter problem are collected hereafter altogether for more clarity, that is,

$$\boldsymbol{\varepsilon} = \mathbf{C}^{-1} : (\boldsymbol{\sigma} - \ell^2 \Delta \boldsymbol{\sigma}) \quad \text{in } V, \quad \partial_n \boldsymbol{\sigma} = \mathbf{0} \quad \text{on } S \tag{38}$$

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{in } V, \quad \mathbf{n} \cdot \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on } S_f \tag{39}$$

$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} \quad \text{in } V, \quad \mathbf{u} = \bar{\mathbf{u}} \quad \text{on } S_c \tag{40}$$

On applying the divergence operator to $(38)_1$ and with the aid of $(39)_1$ and $(40)_1$, we obtain the differential equations

$$\mathcal{L}\mathbf{u} = -\mathbf{b}^*, \quad (\mathbf{b}^* := \mathbf{b} - \ell^2 \Delta \mathbf{b}) \tag{41}$$

already discussed within the Introduction, see (5) and (6). Eq. (41) and the boundary conditions $(39)_2$ and $(40)_2$ constitute a boundary-value problem which, as observed before, is in general coupled with (38).

5.1. Finding the stress field for a given strain field

The integration of the set of PDEs and boundary conditions (38) permits one to evaluate the stress field $\boldsymbol{\sigma}$ associated to a given strain field $\boldsymbol{\varepsilon}$ through only the material constitution. It can be proved that the solution of the latter stress problem admits a variational formulation centered upon the functional $J[\boldsymbol{\sigma}]$ defined as

$$J[\boldsymbol{\sigma}] := \int_V [G(\boldsymbol{\sigma}, \nabla \boldsymbol{\sigma}) - \boldsymbol{\varepsilon} : \boldsymbol{\sigma}] \, \mathrm{d}v \tag{42}$$

where $G(\boldsymbol{\sigma}, \nabla \boldsymbol{\sigma})$ is the Gibbs function (28) and $\boldsymbol{\varepsilon}$ is the assigned strain fields. Namely, it can be proved that *in a stress gradient elastic material, the stress field corresponding to a given strain field, minimizes the functional J; and that conversely the (unique) stress field that minimizes J is the one corresponding to the given strain field.*

The latter statement can be proved by taking the first variation of J, which, after some straightforward mathematical manipulations, can be written as follows

$$\delta J = \int_V [\mathbf{A} : (\boldsymbol{\sigma} - \ell^2 \Delta \boldsymbol{\sigma}) - \boldsymbol{\varepsilon}] : \delta \boldsymbol{\sigma} \, \mathrm{d}v + \int_S \ell^2 \partial_n \boldsymbol{\sigma} : \mathbf{A} : \delta \boldsymbol{\sigma} \, \mathrm{d}a \tag{43}$$

Denoting by $\boldsymbol{\sigma}$ the/a solution to the stress problem (38), then necessarily $\delta J = 0$ for any variation field $\delta \boldsymbol{\sigma}$, hence the functional J is made stationary correspondingly. Conversely, denoting by $\boldsymbol{\sigma}$ the stress solution minimizing J, then δJ must vanish for arbitrary choices of $\delta \boldsymbol{\sigma}$ within $V \cup S$, hence necessarily the field and boundary equations of (38) must be satisfied. On the other hand, let the second variation of J be computed starting from the/a solution to the stress problem. We can write, for any $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + \delta \boldsymbol{\sigma}$,

$$J[\boldsymbol{\sigma}'] = J[\boldsymbol{\sigma}] + \delta J[\boldsymbol{\sigma}] + \frac{1}{2} \delta^2 J[\boldsymbol{\sigma}] \tag{44}$$

where $\delta J[\boldsymbol{\sigma}]$ coincides with (43) and is thus vanishing, whereas $\delta^2 J[\boldsymbol{\sigma}]$ is given by

$$\delta^2 J[\boldsymbol{\sigma}] = 2 \int_V G(\delta \boldsymbol{\sigma}, \nabla \delta \boldsymbol{\sigma}) \, \mathrm{d}v > 0 \tag{45}$$

which is positive for any not trivially vanishing stress variation. Therefore, (44) gives

$$J[\boldsymbol{\sigma}'] = J[\boldsymbol{\sigma}] + \frac{1}{2} \delta^2 J[\boldsymbol{\sigma}] \geqslant J[\boldsymbol{\sigma}] \tag{46}$$

where the equality sign on the right hand side holds if, and only if, $\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma}$; that is, the solution is unique. QED

Note that, since specifying the strain field substantially amounts to specifying the displacement field, then equilibrium considerations are out of concern within the reasoning above.

5.2. Hellinger–Reissner type variational principle for stress gradient elasticity

The equation set (38)–(40) governing the boundary-value problem of stress gradient elasticity admits a unique solution featured by a stationarity principle of the Hellinger–Reissner type (Washizu, 1982). This point is hereafter proved for a stress gradient material featured by a Gibbs function of the type $G(\boldsymbol{\sigma}, \nabla \boldsymbol{\sigma})$ like in (28). The functional to be considered is defined in terms of stress and displacement fields, that is,

$$\begin{aligned} \mathcal{H}[\boldsymbol{\sigma}, \mathbf{u}] := & \int_V [\boldsymbol{\sigma} : \nabla^s \mathbf{u} - G(\boldsymbol{\sigma}, \nabla \boldsymbol{\sigma}) - \mathbf{b} \cdot \mathbf{u}] \, \mathrm{d}v + \int_S \mathbf{r} : \boldsymbol{\sigma} \, \mathrm{d}a \\ & - \int_{S_f} \bar{\mathbf{t}} \cdot \mathbf{u} \, \mathrm{d}a - \int_{S_c} (\mathbf{u} - \bar{\mathbf{u}}) \cdot \mathbf{t} \, \mathrm{d}a \end{aligned} \tag{47}$$

It can be proved that *the solution to the equation set (38)–(40), if it exists, makes $\mathcal{H}$ stationary, and conversely that the stationarity solution solves the mentioned equation set; additionally, the solution in question is unique.*

The first variation of $\mathcal{H}$, applying the divergence theorem and with the aid of some straightforward mathematical manipulations, can be written as follows

$$\begin{aligned} \delta \mathcal{H} = & \int_V [\nabla^s \mathbf{u} - (G_{,\boldsymbol{\sigma}} - \nabla \cdot G_{,(\nabla \boldsymbol{\sigma})})] : \delta \boldsymbol{\sigma} \, \mathrm{d}v - \int_S [\mathbf{n} \cdot G_{,(\nabla \boldsymbol{\sigma})} - \mathbf{r}] : \delta \boldsymbol{\sigma} \, \mathrm{d}a \\ & - \int_V [\nabla \cdot \boldsymbol{\sigma} + \mathbf{b}] \cdot \delta \mathbf{u} \, \mathrm{d}v + \int_{S_f} [\mathbf{n} \cdot \boldsymbol{\sigma} - \bar{\mathbf{t}}] \cdot \delta \mathbf{u} \, \mathrm{d}a \\ & + \int_{S_c} [\mathbf{n} \cdot \boldsymbol{\sigma} - \mathbf{t}] \cdot \delta \mathbf{u} \, \mathrm{d}a - \int_{S_c} [\mathbf{u} - \bar{\mathbf{u}}] \cdot \delta \mathbf{t} \, \mathrm{d}a \end{aligned} \tag{48}$$

Let the pair $(\boldsymbol{\sigma}, \mathbf{u})$, together with the associated strain $\boldsymbol{\varepsilon}(\mathbf{u})$, solve the equation set (38)–(40); then necessarily the square-bracketed expressions of (48) are identically vanishing and thus $\delta \mathcal{H} = 0$ for any variation fields, that is, $\mathcal{H}$ is made stationary correspondingly. Conversely, let $\delta \mathcal{H}$, computed for certain fields $(\boldsymbol{\sigma}, \mathbf{u})$, be vanishing for arbitrary choices of $\delta \boldsymbol{\sigma}$ and $\delta \mathbf{u}$; then necessarily all the square-bracketed expressions of (48) must vanish within the respective domains, which means that the mentioned fields $(\boldsymbol{\sigma}, \mathbf{u})$ together with the associated strain field $\boldsymbol{\varepsilon}(\mathbf{u})$ solve the equation set (38)–(40). The first part of the statement is thus proved.

As for the second part, let us assume that there exist two distinct solutions to the problem (38)–(40), say $\boldsymbol{\sigma}', \boldsymbol{\sigma}''$, etc. Also, let us posit $\breve{\boldsymbol{\sigma}} := \boldsymbol{\sigma}' - \boldsymbol{\sigma}''$, etc., such that the difference fields $\breve{\boldsymbol{\sigma}}, \breve{\mathbf{u}}$ and $\breve{\boldsymbol{\varepsilon}}$ satisfy (38)–(40), but in a homogeneous form. This amounts to stating that $\breve{\boldsymbol{\sigma}}$ is a self-stress field (i.e., in equilibrium with zero body forces in V and zero tractions on S_f) and that $\breve{\boldsymbol{\varepsilon}}$ is a self-strain field (i.e., compatible with zero imposed strains in V and zero displacements on S_c). Additionally, $\breve{\boldsymbol{\sigma}}$ and $\breve{\boldsymbol{\varepsilon}}$ satisfy (38), that is

$$\breve{\boldsymbol{\varepsilon}} = \mathbf{A} : (\breve{\boldsymbol{\sigma}} - \ell^2 \Delta \breve{\boldsymbol{\sigma}}) \quad \text{in } V, \quad \partial_n \breve{\boldsymbol{\sigma}} = \mathbf{0} \quad \text{on } S \tag{49}$$

Then, multiplying $(49)_1$ by $\breve{\boldsymbol{\sigma}}$ and with an integration over V gives, by the (primary) PVP,

$$\int_V \breve{\boldsymbol{\sigma}} : \breve{\boldsymbol{\varepsilon}} \, \mathrm{d}v = \int_V \breve{\boldsymbol{\sigma}} : \mathbf{A} : (\breve{\boldsymbol{\sigma}} - \ell^2 \Delta \breve{\boldsymbol{\sigma}}) \, \mathrm{d}v = 0 \tag{50}$$

This, since $\Delta = \nabla \cdot \nabla$, applying the divergence theorem and with the aid of $(49)_2$, gives the equality

$$\int_V \mathbf{A} :: \{\breve{\boldsymbol{\sigma}} \breve{\boldsymbol{\sigma}} + \ell^2 [(\nabla \breve{\boldsymbol{\sigma}})^{\mathrm{T}} \cdot \nabla \breve{\boldsymbol{\sigma}}]\} \, \mathrm{d}v = 2 \int_V G(\breve{\boldsymbol{\sigma}}, \nabla \breve{\boldsymbol{\sigma}}) \, \mathrm{d}v = 0 \tag{51}$$

That is, the stress difference field makes G vanish, which is possible if, and only if, $\breve{\boldsymbol{\sigma}} \equiv \mathbf{0}$, hence $\breve{\boldsymbol{\varepsilon}} \equiv \mathbf{0}$ and $\breve{\mathbf{u}} \equiv \mathbf{0}$, and thus the solution is unique. QED

Remark 1. A boundary-value problem like (38)–(40) was addressed by Askes and Gutiérrez (2006), whereby $(38)_1$ was derived as an implicit gradient representation of the "nonlocal" strain $\mathbf{e} := \mathbf{C}^{-1} : \boldsymbol{\sigma}$, whereas boundary conditions equivalent to $(38)_2$ were probed via a Lagrange multiplier method. Also, a finite element discretization was provided in which a C^0-continuity of the shape functions is only required.

Remark 2. According to (41), in the case of non-negligible dynamic effects the PDEs (38) take on the form

$$\mathcal{L}\mathbf{u} = \rho\,(\ddot{\mathbf{u}} - \ell^2\Delta\ddot{\mathbf{u}}) - \mathbf{b}^* \tag{52}$$

where ρ is the mass density and $\mathbf{b}^*$ is the enhanced non-inertial body force. Indeed, stress gradient effects manifest themselves also through an enhanced inertia. This is in contrast to the strain gradient model in which gradient inertia effects arise only as a result of an independent modeling of the material inertial behavior (Askes and Aifantis, 2011; Polizzotto, 2012, 2013).

6. The boundary-value problem for strain gradient elasticity

Strain gradient elasticity has received attention since the sixties years (Mindlin, 1965; Mindlin and Eshel, 1968; Germain, 1973; Wu, 1992; Fried and Gurtin, 2006); the set of field and boundary equations governing the related boundary-value problem is well known from the literature. With the same notation used in the previous sections, the latter equations can be written as follows:

$$\boldsymbol{\sigma} = \mathbf{C} : (\boldsymbol{\varepsilon} - \ell^2\Delta\boldsymbol{\varepsilon}) \quad \text{in } V \tag{53}$$

$$\boldsymbol{\tau} = \ell^2\nabla\boldsymbol{\sigma} \quad \text{in } V \tag{54}$$

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}, \quad \boldsymbol{\varepsilon} = \nabla^s\mathbf{u} \quad \text{in } V \tag{55}$$

$$\left.\begin{aligned} &(\overline{\nabla}_{(S)} + H\mathbf{n})\cdot\underbrace{(\mathbf{n}\cdot\boldsymbol{\tau})}_{\Sigma} + \underbrace{\bar{\mathbf{t}} - \mathbf{n}\cdot\boldsymbol{\sigma}}_{\mathbf{t}_{\mathrm{GM}}} = \mathbf{0} \\ &\mathbf{n}\cdot\underbrace{(\mathbf{n}\cdot\boldsymbol{\tau})}_{\Sigma} = \bar{\mathbf{m}} \end{aligned}\right\} \quad \text{on } S_f \tag{56}$$

$$\mathbf{u} = \bar{\mathbf{u}}, \quad \partial_n\mathbf{u} = \bar{\mathbf{g}} \quad \text{on } S_c \tag{57}$$

Here, $\overline{\nabla}_{(S)} := \nabla - \mathbf{n}\partial_n$ denotes the tangential gradient over the surface S at a point where the unit external normal is $\mathbf{n}$ and $H := -\overline{\nabla}_{(S)}\cdot\mathbf{n}$ is twice the mean curvature. On combining the field equations (53) and (55), we can obtain the differential equations

$$\mathcal{L}(\mathbf{u} - \ell^2\Delta\mathbf{u}) = -\mathbf{b} \quad \text{in } V \tag{58}$$

This is a system of fourth-order PDEs in the displacement $\mathbf{u}$, to which the boundary conditions (56) and (57) must be appended. A comparison of (58) with (41) gives a first main difference between the stress gradient model and the strain gradient one. Indeed, in (41) the gradient effects may manifest themselves through the modified body force $\mathbf{b}^*$ (if any), whereas in (58) these effects enter into play through the larger order of the PDEs (fourth order instead of second) with coefficients involving the internal length scale parameter ℓ. But the major differences between the two models come up from the related boundary conditions.

The boundary conditions (56) and (57) have been recently addressed in a study (Polizzotto, 2003, 2012, 2013), where the strain gradient elasticity model was shown to exhibit surface effects as a direct manifestation of the strain-gradient constitution of the material. Namely, whereas the bulk material behaves as a classical Cauchy continuum, instead the material particles close to the boundary surface do coalesce to form up a thin membrane-like boundary layer which obeys the principles of surface mechanics (Gurtin and Murdoch, 1975, 1978), while it confers more rigidity to the elastic system. Indeed, the applied traction $\bar{\mathbf{t}}$ splits in two parts, $\bar{\mathbf{t}} = \mathbf{t}_{\mathrm{GM}} + \mathbf{t}_{\mathrm{C}}$, in which $\mathbf{t}_{\mathrm{C}} = \mathbf{n}\cdot\boldsymbol{\sigma}$ is the part (Cauchy traction) transmitted to the adjacent bulk material, whereas the remaining part, $\mathbf{t}_{\mathrm{GM}}$ (Gurtin–Murdoch traction) is supported by the boundary layer as a surface body force. The boundary layer is in local (and global) equilibrium by its own, under the surface stress $\boldsymbol{\Sigma} := \mathbf{n}\cdot\boldsymbol{\tau}$, the normal component of which equilibrates the applied moment traction $\bar{\mathbf{m}}$ (not permitted for a classical elastic material). The normal derivative of the displacements at the points of S, irrelevant within classical elasticity, do constitute additional degrees of freedom that must be specified on S_c.

The above features exhibited by a strain gradient elasticity model on one hand enrich the mechanics of the inherent behavior and likely make it possible to better capture internal length-scale phenomena; on the other hand, they render computationally more burdening the relevant boundary-value problem (56)–(58) with respect to the classical problem.

We have already commented the equation set (38)–(40). Here, in order to better view the contrast between this equation set and the analogous one for the strain gradient model, we may add the following observations. In contrast to a strain gradient elastic continuum which constitutes a classical Cauchy continuum—in the sense that the inherent Cauchy stress $\boldsymbol{\sigma}$, besides obeying the Cauchy traction theorem, can be computed as the Hookean stress associated to some given strain $\boldsymbol{\varepsilon}$—a stress gradient elastic continuum is *not* a classical Cauchy continuum since the Cauchy stress $\boldsymbol{\sigma}$ associated to a given strain $\boldsymbol{\varepsilon}$ can be obtained only through the integration of a PDE system with the related boundary conditions. Also, in contrast to the boundary layer of a strain gradient elastic material—which is able to support, in all or in part, the external ordinary and moment tractions)—the boundary layer of a stress gradient elastic model is in equilibrium under through-thickness constant stresses and is unable to support, in all or in part, the applied traction $\mathbf{t}$, which in fact is entirely transmitted to the bulk material.

7. The Euler–Bernoulli beam

As an illustrative example, an Euler–Bernoulli beam is considered in both cases of stress gradient and strain gradient constitutive behavior. The beam is referred to Cartesian orthogonal axes with co-ordinates (x, y, z), of which the x axis coincides with the beam longitudinal axis, whereas the plane x–z identifies with the plane of bending. The (uniform) cross section, say Ω, is symmetric with respect to the axis parallel to z. The usual kinematic assumptions for beam models hold, that is, by a standard notation,

$$\left.\begin{aligned} u_x = z\phi(x), \quad u_z = w(x), \quad u_y \equiv 0 \\ \phi = -w'(x), \quad \varepsilon = -zw''(x) \end{aligned}\right\} \tag{59}$$

7.1. Stress gradient beam model. Formulation

In principle, stress gradient effects are allowed to propagate both longitudinally and transversally. However, as it is usually accepted for beam theories, only longitudinal propagation of the latter effects is admitted and therefore the pertinent Gibbs function has the form

$$G = \frac{1}{2E}\left[\sigma^2 + \ell^2(\sigma_{,x})^2\right] \tag{60}$$

The functional of the Hellinger–Reissner principle of Section 5, enforced for the beam model under consideration, can be written as follows:

$$\mathcal{H} = \int_0^L \int_\Omega \{(-zw'')\sigma - G(\sigma, \sigma_{,x}) - p_z w\}\,\mathrm{d}a\,\mathrm{d}x - \underbrace{[\bar{M}\phi + \bar{Q}w]_0^L}_{\text{free ends}} - \underbrace{[(\phi - \bar{\phi})C + (w - \bar{w})P]_0^L}_{\text{constr.ends}} \tag{61}$$

Here, $\bar{M}$ and $\bar{Q}$ denote bending moments and shear forces specified at the *free* ends, whereas $\bar{\phi}$ and $\bar{w}$ denote rotations and deflections assigned at the *constrained* ends where the moment and force reactions C and P act. The stress field σ within the beam volume is free in all, except at the beam end cross sections where it is required to be equivalent to the end couples. The latter constraint is taken in account by writing

$$\int_\Omega z\sigma\,\mathrm{d}a = M_0 \quad \text{at } x = 0, L \tag{62}$$

with M_0 being not subjected to variation. Physically, the latter constraint means to relax the constitutive inability of the material to respond to the stress at the points of the end cross sections, where thus $\sigma_{,x}$ is allowed to be non-vanishing.

Next, on appending the constraint (62) and (61) by the Lagrange multiplier method, we can obtain the new augmented functional

$$\mathcal{H}_a = \int_0^L \int_\Omega \{(-zw'')\sigma - G(\sigma, \sigma_{,x}) - p_z w\}\,\mathrm{d}a\,\mathrm{d}x - \underbrace{[\bar{M}\phi + \bar{Q}w]_0^L}_{\text{free ends}} - \underbrace{[(\phi - \bar{\phi})C + (w - \bar{w})P]_0^L}_{\text{constr. ends}} + \underbrace{\left[\chi\left\{\int_\Omega z\sigma\,\mathrm{d}a - M_0\right\}\right]_0^L}_{\text{all ends}} \tag{63}$$

where $\chi(0)$ and $\chi(L)$ are Lagrange multipliers. After some straightforward mathematics in which the usual definition of the bending moment M is used, that is

$$M := \int_\Omega z\sigma\,\mathrm{d}a \tag{64}$$

and with the notation $p := \int_\Omega p_z \mathrm{d}a$, the first variation of $\mathcal{H}_a$ can be cast as follows:

$$\delta\mathcal{H}_a = -\int_0^L (M'' + p)\delta w\,\mathrm{d}x - \underbrace{[(M - \bar{M})\delta\phi]_0^L}_{\text{free ends}} + \underbrace{[(M' - \bar{Q})\delta w]_0^L}_{\text{free ends}} + \underbrace{[(M - C)\delta\phi]_0^L}_{\text{constr. ends}} + \underbrace{[(M' - P)\delta w]_0^L}_{\text{constr. ends}} + \underbrace{[(\phi - \bar{\phi})\delta C + (w - \bar{w})\delta P]_0^L}_{\text{constr. ends}} - \int_0^L \int_\Omega \left\{zw'' + \frac{1}{E}(\sigma - \ell^2\sigma_{,xx}\right\}\delta\sigma\,\mathrm{d}a\,\mathrm{d}x + \underbrace{\left[\int_\Omega \left(z\chi - \frac{\ell^2}{E}\sigma_{,x}\right)\delta\sigma\,\mathrm{d}a\right]_0^L}_{\text{all ends}} = 0. \tag{65}$$

From the latter equation we obtain, besides the identifications $M = C$ and $M' = P$ at the constrained ends, the beam governing field equations, that is,

$$\left.\begin{aligned} M'' + p &= 0 \\ -Ezw'' &= \sigma - \ell^2\sigma_{,xx} \end{aligned}\right\} \quad (0 < x < L) \tag{66}$$

and the boundary conditions

$$\left.\begin{aligned} &M = \bar{M}, \quad M' = \bar{Q} \quad \text{at the free ends} \\ &\phi = \bar{\phi}, \quad w = \bar{w} \quad \text{at the constr. ends} \end{aligned}\right\} \tag{67}$$

A further identification is also obtained as

$$z\chi(\xi) = \frac{\ell^2}{E}\sigma_{,x}|_{x=\xi} \quad \text{for } \xi = 0, L \tag{68}$$

or equivalently

$$\chi(\xi) = \frac{\ell^2}{EI}M'(\xi) \quad \text{for } \xi = 0, L. \tag{69}$$

Remark 3. Eq. (69) explains the necessity for introducing the constraint (62). Without the latter constraint it has to be $\chi(0) = \chi(L) = 0$; then, as shown by (69), the variational procedure would lead to $Q(0) = Q(L) = 0$, which implies that the beam should displace freely at the ends. Indeed, no meaningful beam theory would exist in the absence of the mentioned constraint.

7.2. Solution of the stress gradient beam model

By $(66)_2$, multiplying by z and with an integration over Ω, we can write, using $(66)_1$,

$$M = -EIw'' - \ell^2 p \tag{70}$$

which by a further double differentiation gives

$$EIw'''' = p - \ell^2 p'' \tag{71}$$

Therefore, denoting by $f = f(x)$ a particular function such that $f'''' = p$, the general solution of (71) is of the form

$$EIw(x) = f - \ell^2 f'' + C_1x^3 + C_2x^2 + C_3x + C_4 \tag{72}$$

where C_1, C_2, C_3, C_4 are some constants. Then, by (70) we have

$$M = -f'' - (6C_1x + 2C_2) \tag{73}$$

The constants C_1, C_2, C_3, C_4 can be determined by the (standard) boundary conditions (67), hence both the w and M functions can be considered known.

It remains to evaluate the stress field by $(66)_2$ which by (70) is equivalent to the following differential equation

$$\sigma - \ell^2\sigma_{,xx} = \frac{z}{I}(M + \ell^2 p) \tag{74}$$

The general solution of the latter equation is

$$\sigma = B_1(y,z)\cosh\frac{x}{\ell} + B_2(y,z)\sinh\frac{x}{\ell} + \frac{zM(x)}{I} \tag{75}$$

In order that, in every cross section of abscissa x, the latter stress field be statically equivalent to a couple of the plane x–z of intensity $M(x)$, the coefficients $B_1(y,z)$ and $B_2(y,z)$ must satisfy the conditions

$$\left.\begin{aligned} &\textstyle\int_\Omega B_1(y,z)\,\mathrm{d}a = \int_\Omega B_2(y,z)\,\mathrm{d}a = 0 \\ &\textstyle\int_\Omega yB_1(y,z)\,\mathrm{d}a = \int_\Omega yB_2(y,z)\,\mathrm{d}a = 0 \\ &\textstyle\int_\Omega zB_1(y,z)\,\mathrm{d}a = \int_\Omega zB_2(y,z)\,\mathrm{d}a = 0 \end{aligned}\right\} \tag{76}$$

Moreover, according to (68) and (69), the following restrictions must be satisfied, i.e.,

$$\left.\begin{aligned} &\sigma_{,x}|_{x=L} = \tfrac{B_1}{\ell}\sinh\tfrac{L}{\ell} + \tfrac{B_2}{\ell}\cosh\tfrac{L}{\ell} + z\tfrac{M'(L)}{I} = z\tfrac{E}{\ell^2}\chi(L) \\ &\sigma_{,x}|_{x=0} = \tfrac{B_2}{\ell} + z\tfrac{M'(0)}{I} = z\tfrac{E}{\ell^2}\chi(0) \end{aligned}\right\} \tag{77}$$

at all points (y,z) within Ω. By (69) and (77) transforms into

$$\frac{B_1}{\ell}\sinh\frac{L}{\ell} + \frac{B_2}{\ell}\cosh\frac{L}{\ell} = 0, \quad \frac{B_2}{\ell} = 0 \tag{78}$$

which leads to the conclusion that $B_1 = B_2 \equiv 0$, and that therefore the stress σ takes on the classical Navier form, that is,

$$\sigma = \frac{zM(x)}{I} \tag{79}$$

In conclusion, we can observe that, within the considered beam model, the stress gradient effects manifest themselves through the beam deflection curve $w(x)$ if, and only if, the applied load is distributed in a nonlinear fashion, such that $p'' \neq 0$. Moreover, the latter effects manifest themselves also through the bending moment and the stress fields if in addition the beam is statically indeterminate.

7.3. The strain gradient beam model

We may start the discussion on the strain gradient beam model with a formulation analogous to the one of Section 7.1; for simplicity we skip this point and proceed by deriving the essential concepts from the governing equations (53)–(57) related to a strain gradient continuum. Indeed, the same basic kinematics of (59) holds. As in the case of the stress gradient beam model, here we also assume that strain gradient effects can only propagate longitudinally. Two types of stresses are essential, that is, the (total) stress σ (force per unit area) and the higher order (or double) stress τ (moment per unit area, or force per unit length), which are related to the strain by the following constitutive equations

$$\left.\begin{aligned} \sigma &= E(\varepsilon - \ell^2\varepsilon'') = -Ez(w - \ell^2 w'')'' \\ \tau &= \ell^2 E\varepsilon' = -\ell^2 Ezw''' \end{aligned}\right\} \tag{80}$$

These are the beam counterparts of (53) and (54), respectively. The latter stresses generate two types of bending moments, that is,

$$M = \int_\Omega z\sigma\,\mathrm{d}a, \quad \mathcal{M} = \int_\Omega z\tau\,\mathrm{d}a \tag{81}$$

which, referred to as the ordinary and the higher order (or double) bending moments, are work-conjugate of ϕ and ϕ', respectively. Their constitutive equations are readily found as

$$M = -EI(w - \ell^2 w'')'', \quad \mathcal{M} = -\ell^2 EIw''' \tag{82}$$

The beam equilibrium equations remain the classical ones together with the standard boundary conditions. The higher order boundary condition $(56)_2$ and $(57)_2$ can be translated as follows

$$\left.\begin{aligned} &\mathcal{M} = -\ell^2 EIw''' \rightarrow \text{assigned at the free ends} \\ &\phi' = -w'' \rightarrow \text{assigned at the constr. ends} \end{aligned}\right\} \tag{83}$$

Next, by $(82)_1$ we can write

$$M = -EI(w - \ell^2 w'')'' \quad (0 < x < L) \tag{84}$$

Then, with a further double differentiation and using the beam equilibrium equation, we obtain

$$(w - \ell^2 w'')'''' = p/EI \tag{85}$$

which amounts to

$$w - \ell^2 w'' = f(x)/EI + C_1x^3 + C_2x^2 + C_3x + C_4 \tag{86}$$

where C_1, C_2, C_3, C_4 are constants and f is a particular function such that $f'''' = p$. The general solution of the latter differential equation can be written as

$$\begin{aligned} w(x) = {} & A_1\cosh\frac{x}{\ell} + A_2\sinh\frac{x}{\ell} + F(x)/EI + C_1x^3 + C_2x^2 \\ & + (C_3 + 6\ell^2 C_1)x + C_4 + 2\ell^2 C_2 \end{aligned} \tag{87}$$

where A_1, A_2 are some further constants and F is a particular function satisfying the equation

$$F - \ell^2 F'' = f. \tag{88}$$

The six constants $C_1, C_2, C_3, C_4, A_1, A_2$ can be evaluated by means of the four standard boundary conditions and the two nonstandard higher order ones (83). Therefore both the deflection and the bending moments diagrams can be considered known. The stress distribution has to be computed by the Navier formula $\sigma = zM/I$.

We can conclude stating that in a strain gradient beam model the strain gradient effects manifest themselves through the beam deflection curve no matter how the beam is loaded, and that these effects may emerge also through the bending moment and stress fields provided the beam is statically indeterminate.

On comparing the two beam models the following can be stated:

(i) A stress gradient beam subjected to a piecewise linearly distributed load behaves as a classical beam since no stress gradient effects can manifest themselves. If instead the applied load is piecewise nonlinear, then the stress gradient effect do manifest themselves through the beam deflection and, if the beam is statically indeterminate, also through the bending and stress fields. In the case of dynamic loading, induced gradient inertia effects do emerge.

(ii) In a strain gradient beam, the strain gradient effects manifest themselves through the beam deflection independently of the load conditions, but—like in the case of a stress gradient beam—the latter effects emerge also through the bending and stress fields only if the beam is statically indeterminate. In the case of dynamic loading, gradient inertia effects manifest themselves only as a result of an *ad hoc modeling of the material inertia behavior.*

Another remark suggested by the preceding comparisons is that—at parity of data—the two constitutive models may lead to responses notably different from each other. This fact has been exemplified by addressing a clamped–clamped beam of length L subjected to a sinusoidal load, say $p(x) := p_0 \cos \pi x/L$, $-L/2 \leqslant x \leqslant L/2$. The respective solutions are as follows:

For the stress gradient beam:

$$w(x) = \frac{p_0 L^4}{\pi^3 EI} Q \left\{ \frac{1}{\pi}\cos\frac{\pi x}{L} - \frac{1}{4}\left[1 - \left(\frac{2x}{L}\right)^2\right] \right\} \tag{89}$$

$$M(x) = \frac{2p_0 L^2}{\pi^3}\left[\frac{\pi}{2}\cos\frac{\pi x}{L} - Q\right] \tag{90}$$

For the strain gradient beam:

$$w(x) = \frac{p_0 L^4}{\pi^3 EI}\frac{1}{Q}\left\{\frac{1}{\pi}\cos\frac{\pi x}{L} - \frac{1}{Q_1}\Phi(x)\right\} \tag{91}$$

$$M(x) = \frac{2p_0 L^2}{\pi^3}\left[\frac{\pi}{2}\cos\frac{\pi x}{L} - \frac{1}{QQ_1}\right], \tag{92}$$

where

$$\Phi(x) := \frac{1}{4}\left[1 - \left(\frac{2x}{L}\right)^2\right] - \frac{1}{2}\left(\frac{2\ell}{L}\right)^2\left(1 - \frac{\cosh\frac{x}{\ell}}{\cosh\frac{L}{2\ell}}\right) \tag{93}$$

and

$$Q := 1 + \left(\frac{\pi\ell}{L}\right)^2, \quad Q_1 := 1 - \frac{2\ell}{L}\tanh\frac{L}{2\ell} \tag{94}$$

Fig. 1 illustrates how the latter solutions change with the internal length parameter $\lambda := \ell/H$, where H is a linear dimension of the cross section, for instance $H = I^{1/4}$.

In Fig. 1(a), the displacement (or deflection) of the beam mid section, normalized by taking equal to one its value for the classical beam, is plotted as a function of the internal length parameter, say

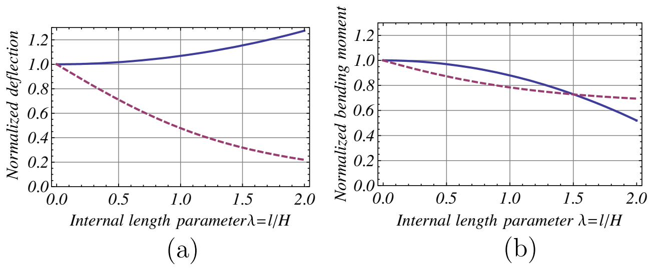

Fig. 1. Response of a clamped–clamped beam to a sinusoidal load for stress gradient (solid lines) and strain gradient (dashed lines) constitutive behavior: normalized deflection (a) and normalized bending moment (b) at the beam mid cross section.

$\lambda = \ell/H$, for the stress gradient (solid line) and for the strain gradient (dashed line) material models; in Fig. 1(b), the bending moment at the central cross section is analogously plotted. We can observe the different manners in which gradient effects manifest themselves in the two beam models.

Fig. 1(a) shows that, under constant load, the deflection of the beam increases with increasing λ in the case of stress gradient material, whereas it decreases in the case of strain gradient material. These behaviors are quite reasonable considering that (i) within stress gradient models the stress gradient effects enter into play through an enhanced load (which in the present case is expressed as $(1+Q)p$); and (ii) within strain gradient models, on one hand, the acting load remains unaffected by λ, on the other hand the gradient effects enter into play through the strain gradient.

Fig. 1(b) shows that, under constant load, the normalized bending moment of the beam mid section decreases with increasing λ for both models, but with opposite rates.

8. Conclusions

Inspired by a landmark work by Eringen (1983), we have addressed a class of stress gradient elasticity models within a thermodynamics framework, whereby the Eringen stress gradient constitutive equations have been determined together with the related (so far uncertain) boundary conditions. A Hellinger–Reissner type variational principle has been devised to characterize the (unique) solution of the relevant boundary-value problem. The complementary characters of the stress gradient model with respect to the strain gradient one have been pointed out and a comparison between each other has been accomplished also considering an Euler–Bernoulli beam model. The original contributions provided by the present paper can be enumerated as follows:

1. Formulation of an extended complementary principle of the virtual power (PVP) for stress gradient materials, by which the pertinent higher order field and boundary compatibility equations (counterparts of the analogous higher order equilibrium equations for strain gradient materials) have been determined.
2. Formulation of a complementary form of the Clausius–Duhem inequality for the assessment of the restrictions of the constitutive equations of a material possessing a Gibbs free enthalpy (function of the stress, the stress gradient and the temperature). These restrictions include, besides the Eringen stress–strain differential equations, the related higher order boundary conditions in the Neumann form $\partial_n \sigma_{ij} = 0$ at all points of the boundary surface.
3. Formulation of variational principles for stress gradient elasticity, that is, (i) Minimum principle for the stress response of a stress gradient material subjected to a specified strain field; (ii) Hellinger–Reissner type stationarity principle in terms of stresses and displacements characterizing the solution of the relevant boundary-value problem.
4. Comparison of the Eringen stress gradient elasticity model with the analogous (Aifantis) strain gradient one, of which the main differences are pointed out in relation to their mechanical behaviors (and in particular to the relevant surface effects), as well as to the computational aspects.
5. Exploitation of the Hellinger–Reissner principle mentioned above to derive a fully consistent theory of stress gradient Euler–Bernoulli beams, with an assessment of the sensitivity of the beam response to the stress gradient effects in comparison to the analogous strain gradient beam model. The latter beam theory explains the reasons why the higher order boundary conditions are relaxed and why the stress is given by the classical Navier formula. This theory collects and unifies a number of results and notions already known from the wide literature; its extension to dynamics and buckling conditions is straightforward.

The author believes that the present paper provides valuable contributions to the knowledge of the theoretical foundations of stress gradient elasticity. Further study to include dynamics and other aspects of the material behavior will be pursued in the near future.

References

Aifantis, E.C., 1992. On the role of gradients in the localization of deformation and fracture. Int. J. Eng. Sci. 30, 1279–1299.
Altan, B.S., Aifantis, E.C., 1997. On some aspects of the special theory of gradient elasticity. J. Mech. Behav. 8, 231–282.
Askes, H., Aifantis, E.C., 2011. Gradient elasticity in statics and dynamics: an overview of formulations, length scale identification procedures, finite element implementations and new results. Int. J. Solids Struct. 48, 1962–1990.
Askes, H., Gutiérrez, M.A., 2006. Implicit gradient elasticity. Int. J. Numer. Meth. Eng. 67, 400–416.
Eringen, A., 1983. On differential equations of nonlinear elasticity and solutions of screw dislocation and surface waves. J. Appl. Phys. 54, 4703–4710.
Eringen, A.C., 2002. Nonlocal Continuum Field Theories. Springer, New York.
Fried, E., Gurtin, M.E., 2006. Tractions, balances, and boundary conditions for nonsimple materials with applications to liquid flow and small-length scales. Arch. Ration Mech. Anal. 182, 513–554.
Germain, P., 1973. The method of the virtual power in continuum mechanics. SIAM Appl. Math. 25, 556–575.
Gurtin, M.E., Murdoch, I., 1975. A continuum theory of elastic material surface. Arch. Ration Mech. Anal. 57, 291–323.
Gurtin, M.E., Murdoch, I., 1978. Surface stresses in solids. Int. J. Solids Struct. 14, 431–440.
Kumar, D., Heinrich, C., Waas, A.M., 2008. Buckling analysis of carbon nanotubes using nonlocal continuum theories. J. Appl. Phys. 103 (1–8), 073521.
Lazar, M., Maugin, G.M., Aifantis, E.C., 2006a. On a theory of nonlocal elasticity of bi-Helmoltz type and some applications. Int. J. Solids Struct. 43, 1404–1421.
Lazar, M., Maugin, G.M., Aifantis, E.C., 2006b. Dislocations in second strain gradient elasticity. Int. J. Solids Struct. 43, 1787–1817.
Lemaitre, J., Chaboche, J.-L., 1990. Mechanics of Solid Materials. Cambridge University Press, Cambridge.
Maugin, G.A., 1999. The Thermodynamics of Nonlinear Irreversible Behaviors. World Scientific Publishing Co., Pte. Ltd..
Mindlin, R.D., 1965. Second gradient of strain and surface-tension in linear elasticity. Int. J. Solids Struct. 1, 417–438.
Mindlin, R.D., Eshel, N.N., 1968. On first strain-gradient theories in linear elasticity. Int. J. Solids Struct. 28, 845–858.
Peddieson, J., Buchanan, G.R., McNitt, R.P., 2003. Application of nonlocal continuum models to nanotechnology. Int. J. Eng. Sci. 41, 305–312.
Polizzotto, C., 2003. Gradient elasticity and nonstandard boundary conditions. Int. J. Solids Struct. 40, 7399–7423.
Polizzotto, C., 2011. A unified residual-based thermodynamic framework for strain gradient theories of plasticity. Int. J. Plast. 27, 388–413.
Polizzotto, C., 2012. A gradient elasticity theory for second-grade materials and higher order inertia. Int. J. Solids Struct. 49, 2121–2137.
Polizzotto, C., 2013. A second strain gradient elasticity theory with second velocity gradient inertia—Part I: Constitutive equations and quasi-static behavior, Part II: Dynamic behavior. Int. J. Solids Struct. 50, 3749–3765 (3766–3777).

Polizzotto, C., Pisano, A.A., 2012. An energy-residual based approach to gradient effects within the mechanics of generalized continua. J. Mech. Behav. Mater. 21 (3–4), 101–121.

Reddy, J.N., 2007. Nonlocal theories for bending, buckling and vibration of beams. Int. J. Eng. Sci. 45, 288–307.

Reddy, J.N., Pang, S.D., 2008. Nonlocan continuum theories of beams for the analysis of nanotubes. J. Appl. Phys. 103, 023511.

Ru, C.Q., Aifantis, E.C., 1993. A simple approach to solve boundary-value problems in gradient elasticity. Acta Mech. 101, 59–68.

Washizu, K., 1982. Variational Methods in Elasticity and Plasticity, Third Edition. Pergamon Press, New York.

Wu, C.H., 1992. Cohesive elasticity and surface phenomena. Q. Appl. Math. 1, 73–103.